ACTES DE LA 13e CONFÉRENCE INTERNATIONALE

# PSYCHOLOGY OF MATHEMATICS EDUCATION

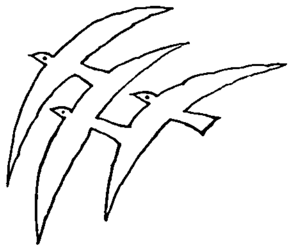




Paris (France)

9-13 juillet 1989

P.M.E. 13



***Bicentenaire de la Révolution Française***

Volume 3

ACTES DE LA 13EME CONFERENCE INTERNATIONALE

# PSYCHOLOGY OF MATHEMATICS EDUCATION

Paris, France

9-13 Juillet 1989

P M E 1 3

Volume 3

Edité par

Gérard Vergnaud
Janine Rogalski
Michèle Artigue
G.R. DIDACTIQUE
CNRS - PARIS V
Laboratoire PSYDEE
46, rue Saint-Jacques
F. 75005 PARIS

# A MODEL OF UNDERSTANDING TWO-DIGIT NUMERATION AND COMPUTATION

Hanlie Murray and Alwyn Olivier
University of Stellenbosch

*This paper suggests that a full understanding of place value is not a prerequisite for many powerful, pliable computational strategies; that these strategies are formulated and widely used by young students; and that the use of these computational strategies facilitates a full understanding of place value. Based on an analysis of the computational strategies employed by young children, a model is proposed for the development of children's understanding of two-digit numbers. This model holds serious implications for both when and how to introduce two-digit numbers, and also for the role assigned to standard written algorithms in the junior school.*

## INTRODUCTION

Various models have been proposed to describe the development of young children's understanding of place value, e.g. Resnick (1983), Kamii (1985), Ross (1985). The general consensus among these researchers and also authors such as Richards and Carter (1982) seems to be that a full understanding of place value necessitates the conceptualization of *ten* as a new abstracted repeatable (iterable) unit which can be used as a unit to construct other numbers. It is also clear that this abstraction is quite difficult and that many third and fourth grade children have not attained this understanding in spite of many years of intensive teaching about place value.

We have available taped and transcribed protocols of interviews conducted at the *start* of the school year in 1987 with all 140 third grade pupils (aged eight to nine) of two fairly representative white schools. During each interview the student was presented with context-free addition problems involving whole numbers of increasing size, first orally, then set out horizontally, and finally set out vertically with the digits correctly aligned. The student was encouraged to solve each problem in whatever way he chose to, and asked to describe his solution strategy for every problem. These students had all had at least nine months' intensive instruction in place value and the standard vertical algorithm for addition. Analysis of the protocols shows, however, that these children use the taught algorithm infrequently, but rather prefer informal (untaught) computational strategies. The data also show vast qualitative differences in understanding of two-digit numbers, that is evidenced by the different types of computational strategies utilized by different students. The data have led us to postulate four levels of understanding of two-digit numbers, each level easily identified by the computational strategies employed to perform context-free computations. The levels that *precede* a full understanding of place value are probably mportant and far more useful than has been realized before, and function as vital pmental stepping stones towards the place value concept.

## DIFFERENT RANGES OF NUMBERS

Steffe, Von Glasersfeld, Richards and Cobb (1983) describe children's understanding of the single-digit numbers as progressing through different levels of abstraction until the number is constituted as an abstract unit item with a meaning independent of physical objects or counting acts. This implies acquiring the *numerosities* ("how-manyness") of the numbers. Whereas children who have not yet acquired the numerosities of the range of numbers with which they have to perform computations must necessarily utilize pre-numerical strategies like counting all, children who *have* acquired the numerosities of the numbers have the capacity to use numerical strategies like counting on from first, counting on from larger, etc.

One could expect, maybe, that children's use of numerical strategies in computations with small numbers will transfer to computations with larger numbers. It has, however, frequently been documented that children find problems involving smaller numbers easier than those involving larger numbers (e.g. Carpenter and Moser, 1982), and that children change their behaviour when the sizes of the numbers in a given situation change (Cooper, 1984). The following are clear examples from our research showing children regressing to more primitive strategies or completely senseless juggling of symbols when they have to compute with larger numbers. (These problems were all presented orally. A summary of the child's strategy is given next to the problem.)

**Elsa – a regression to the pre-numerical strategy of counting all**

| | |
|---|---|
| 7 + 5 = 12 | 5 + 6 = 11 and add 1 |
| 9 + 2 = 11 | 9 + 1 = 10; + 1 = 11 |

but

| | |
|---|---|
| 11 + 17 = 27 | draws 11 small circles, then 17 small circles, then counts all |
| 37 + 5 = 42 | draws 37 small circles, then 5 small circles, then counts all |

**Marlene – a regression to meaningless manipulation of digits**

| | |
|---|---|
| 9 + 2 = 11 | 9 + 1 = 10; 9 + 2 = 11 |
| 9 + 6 = 15 | 9 + 1 = 10; + 5 = 15 |

but

| | |
|---|---|
| 29 + 4 = 96 | writes 29 + 4, then: 2 + 4 = 6 and puts 9 next to the 6 |
| 23 + 12 = 53 | writes 23 + 12, then: 2 + 3 = 5 and 1 + 2 = 3 |
| 25 + 8 = 87 | writes 25 + 8, then: 2 + 5 = 7 and puts the 8 next to the 7 |

We argue that this regression is explained by the fact that these numbers are outside the children's range of constructed numerosities (and in Marlene's case, coupled with a perspective of mathematics as meaningless manipulation of meaningless symbols). When a child has acquired the numerosities of the smaller numbers, e.g. up to nine or twelve, he has not necessarily acquired the numerosities of the two-digit numbers as well, e.g. it is clear that Elsa's lack of "feeling" for 37 forces her to *recreate* 37 by means of circles which can be counted from the beginning. Although a child may therefore be able to employ numerical strategies within a *certain* range of numbers, the numerosities of numbers beyond this range have also to be acquired before he is capable of using numerical strategies when computing in a range of larger numbers.

## DIFFERENT TYPES OF COMPUTATIONAL STRATEGIES

When children work with smaller numbers, their computational strategies fall into two broad classes: the *pre-numerical* strategies where the child has to count all because he has not yet acquired the numerosities of the numbers he is using, and the *numerical* strategies like counting on or bridging through ten. In computations with two-digit numbers, the pre-numerical/numerical distinction between strategies of course still exists. We can also distinguish different types of *numerical* strategies.

One type of numerical strategy is counting on. Another type, not based on counting, Carpenter (1980) calls heuristic strategies. Heuristic strategies often involve the decomposition of one or more of the numbers in a problem in order to transform the given problem to an easier problem or series of problems, e.g.

$$36 + 27 = 36 + 4 + 23 = 40 + 23 = 40 + 20 + 3 = 60 + 3.$$

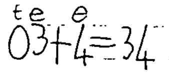
Peter solves 36 + 27 as: "Three *tens* and two *tens* gives fifty, and six and then seven, which gives 63," whereas Marietjie solves the same problem by saying "*Thirty* and *twenty* gives fifty, then add six and seven." Although seemingly the same strategy, we see the different naming as manifestation of different understandings of two-digit numeration.

## THE MODEL

We hypothesize that there is a relationship between children's understanding of two-digit numbers and the computational strategies that they use. It is not necessarily a linear relationship, because children do not consistently use their optimal computational strategies; at best we can say that the use of a certain type of computational strategy "defines" a certain minimal understanding of number and numeration. Based on our research data and a theoretical conceptual analysis, we have formulated a theoretical model describing four increasingly abstract levels of types of computational strategies with two-digit numbers in a given range, each type associated with its prerequisite understanding of number and numeration.

### The first level

At the first level the child has not yet acquired the numerosities of two-digit numbers in a given range, and can therefore only use the pre-numerical strategy of counting all for computations in this range. The child knows the number names of the two-digit numbers and their associated numerals, and associates the whole numeral with the number it represents, but assigns no meaning to the individual digits. At this level the symbol group 63 can be regarded as a way of "spelling" the number name. A common error is to interchange the digits (e.g. writing 36 for sixty-three), yet often this has no adverse effect on the child's understanding of the number itself, as evidenced by Johan. To solve 30 + 4 (presented orally) he writes:




and says "thirty plus four is thirty-four." His incorrect use of the t and e symbols (for "tens" ... nits" in Afrikaans) in no way affects his computation, because the "tens" and "units" ... o meaning for him yet.

Results of previous research support the conclusion that the understanding of the whole numeral precedes understanding of the individual digits (e.g. Barr, 1978; Kamii, 1986).

### The second level

At this level the child has acquired the numerosities of the two-digit numbers in a given range, which implies that he can utilize numerical computational strategies like counting on for computations in that range.

Whereas it is sufficient for a child to use only counting on strategies for smaller numbers, counting on becomes very tedious and also prone to error when used with larger two-digit addends.

### The third level

At this level the child sees a two-digit number as a composite unit, and can decompose or partition the number into other numbers that are more convenient to compute with, e.g. to replace 34 with 30 and 4. This provides the child with the conceptual basis to use heuristic strategies.

The heuristic strategies used by students in our research are almost always based on *decimal* decomposition, i.e. a decomposition into a multiple of ten and some units, e.g. 67 as 60 + 7. But the tens are most emphatically *not* treated as "so many tens"; they are called by their *full number names*, e.g. sixty-seven becomes "a sixty and a seven", *not* "six *tens* and seven *units*." Students then use their knowledge of adding multiples of ten to obtain answers, e.g. Chris does 23 + 12 by saying: "Take the three away, add the twelve to the twenty, then add the three again", partitioning 23 into twenty and three. If both numbers are large, he partitions both: 36 + 27 is solved as "take the six and the seven away, thirty plus twenty is fifty; now add six, then add seven."

We have identified ten different heuristic strategies based on decimal decomposition. These strategies are very powerful and completely trustworthy: we have not found children applying them incorrectly. If the answer is incorrect, it has always been because the child has failed to add the units correctly.

### The fourth level

At this level the child is truly able to think of a two-digit number as consisting of *groups* of tens and some units, i.e. the child can conceptualize *ten* as a new iterable unit, without losing the meaning of the number as a number. Whereas at level 3 the child works with ten as a *number*, that is no different than any other number, at level 4 he is able to work with ten as an *iterable unit*, a thing that can be counted as a unit, so that e.g. the number 23 is conceptualized as "two tens and three ones." Richards and Carter (1982) make this distinction clear:

> "Seeing ten as iterable is distinct form (*sic*) being able, say, to add ten and ten to make twenty. Seeing twenty as built up out of two units of ten is conceptually different from simply being able to add ten and ten to get twenty. In this sense, 'Ten and Ten' are distinct from 'Two Tens'. The former is not different from taking any pair of numbers..." (p. 61)

Concerning computation, level 4 understanding of numeration facilitates a progressive schematization ("shortening") and abstraction of the level 3 heuristic strategies. Here are some examples of this type of "groups of ten" thinking as opposed to the "tens part as a complete number" thinking of the previous level: Annemarie solves 26 + 37 (presented vertically) by saying: "Six plus seven is thirteen. Five tens plus one ten is sixty. Sixty plus three is sixty-three." For 36 + 27 she says: "Thirty plus two tens, that's fifty. Six plus seven is thirteen, that's sixty-three." Very few of the children we interviewed use this conceptualization of ten as a new "unit"; even Annemarie frequently prefers the level 3 method:

$$39 + 14 \rightarrow 30 + 10 = 40;\ 4 + 9 = 13;\ 40 + 10 + 3 = 53.$$

Level 4 understanding of numeration is a prerequisite for the *meaningful* execution of the standard written algorithms. A further abstraction allows one to operate on the *digits* of numbers—e.g. in the number 56, the meaning of the five as fifty or five tens can *temporarily* be suspended to work with 5 as a digit for the sake of convenience and the further progressive schematization of computational strategies.

When executing the standard algorithm has become automatic, it is difficult to deduce from the child's behaviour his understanding of the procedure and the underlying numeration concepts. However, it is clear from our research that many children *seem* to think of "groups of tens" in the correct (meaningful) way, because they *talk* about tens and units while they are computing, yet closer examination reveals completely superficial use of the terms "tens" and "units", with no possible evaluation of the numbers involved and the acceptability of the answers obtained. Sonja shows a proficiency with the standard algorithm for vertical addition, which is yet *not* based on a true understanding of the number symbols. She computes 34 + 17 and even 26 + 37 successfully by means of the standard vertical addition algorithm, but 5 + 37 (also written vertically with the digits aligned correctly) as 5 + 3 + 7 = 15, and 5 + 23 becomes 55 (the first 5 becomes the tens of the answer, and the units of the answer are the sum of 2 and 3). A superficial facility in executing the standard written algorithms may therefore hide serious deficiencies in the understanding of numbers and place value.

We have not come across a single child who operates with level 3 strategies ("the tens part as a *number*," e.g. "sixty") showing confusion of the above kind, probably because the mathematics underlying the level 3 strategies can never be hidden from the child: it is impossible to employ a level 3 strategy without understanding what you are doing, but it is extremely easy to implement a standard written algorithm in rote fashion. When the standard written algorithm is routinely employed, one operates on the *syntactic* level, manipulating the symbols *directly* as 'concrete' objects of thought according to certain rules, totally removed from their meanings as *numbers*. The level 3 and 4 heuristic strategies are, however, on the *semantic* level: One deals with the symbols by referring back to their meaning, i.e. in 23 the 2 refers to 20 or two tens. Many students who falter using the standard algorithm either do not have the necessary level 3 semantic knowledge to monitor their syntactic rules, or their syntactic and semantic knowledge appear to co-exist completely unconnected.

## SOME RESULTS

The following table represents a summary of a preliminary analysis of the protocols of a few selected computations:

Percentage of students using types of strategies[a] [b]

| Computation | | Levels 1 & 2 Counting strategies | Levels 3 & 4 Heuristic strategies | Standard algorithm[c] |
|---|---|---|---|---|
| 25 + 8 | (set orally) | 39 (65) | 41 (81) | 9 (46) |
| 21 + 8 | (set horizontally) | 34 (67) | 44 (82) | 11 (25) |
| 27 + 6 | (set vertically) | 31 (81) | 36 (92) | 22 (16) |
| 34 + 21 | (set orally) | 16 (22) | 49 (77) | 16 (55) |
| 34 + 23 | (set horizontally) | 11 (27) | 56 (90) | 19 (63) |
| 36 + 27 | (set orally) | 10 (7) | 42 (73) | 22 (19) |
| 26 + 37 | (set vertically) | 4 (17) | 34 (87) | 21 (31) |

[a]Students not included in this summary either 'did not know', 'guessed', 'knew', or were not asked, because they failed or persevered with similar strategies in similar problems.

[b]Numbers in parenthesis represent the percentage of students who used a particular type of strategy that solved the problem correctly.

[c]A student was coded as using the standard algorithm if he gave direct written or verbal evidence of computing ones and tens separately as digits, from right to left.

The data clearly show to what extent students prefer heuristic strategies, and the high success rate of these strategies. In contrast, the data also show how few students actually employ the standard taught algorithm, as well as the low success rate in using the algorithm. The data also show, however, that a large number of students could not cope with the computations at all (e.g. in the last two categories a maximum of 26% and 41% respectively).

## DISCUSSION

If our model provides an accurate description of the development of children's understanding of two-digit numeration, and if one believes that instruction should be based on the developmental sequences observed in children, then the model and our data have serious implications for the *teaching* of two-digit numeration and computation.

We stress that our subjects have had intensive instruction in "tens and units" place value and in the standard written algorithm for addition. While it is acknowledged that this type of instruction had contributed to the facility of many students with heuristic strategies (that were *not explicitly taught*, and that they preferred to the standard algorithm), this type of instruction also contributed to some students regressing to primitive (but to them meaningful) counting strategies for computing with larger numbers, to students' poor grasp of the standard algorithm when they chose to use it, and the helplessness of many others.

The near universal method of introducing two-digit numeration is by quantifying sets of objects by groupings of tens and ones and learning the numeral and number name associated with the sets of tens and ones. This approach is based on an *a priori* logical analysis of the concepts and has a great deal of intuitive appeal (to teachers) because of the understanding that (supposedly) precedes the symbolization. Yet, this approach does not consider the psychological nature of children's learning: understanding of two-digit numbers as groups of tens and ones is at level 4 and can therefore be expected to be too abstract for students who are operating at level 1, 2, or 3. We have ample evidence that it is not successful to teach children about the tens and ones meaning of the symbols in the symbol groups *before* they have become accustomed to a symbol group as representing a single number (level 1). The child has to work with 63 as a way of writing "sixty-three" for a *long* time before he becomes ready to understand 63 as 6 tens and 3 ones. Similarly, level 2 and level 3 thinking are necessary prerequisites for children to understand the sophistication of two-digit numeration and computation (cf. Murray, 1988).

There is some evidence that the compositional structure of numbers arises first in the context of oral counting. Kamii (1985, 1986) attributes children's difficulty with place-value partly to the teaching of standard procedures and outlines a teaching sequence based on counting, and reading and writing numerals without groups of tens, and on children inventing their own procedures to add 2–, 3– and 4-digit numbers. In a teaching experiment Barr (1978) found that kindergarten children who were introduced to two-digit numeration through counting, and reading and writing numerals before grouping exercises designed to provide understanding, did better than those who did the grouping exercises first.

It it seems that when students' level 1 and 2 counting strategies become too cumbersome for computation with larger numbers, teachers "help" children by introducing the standard algorithms as necessary (the only) computational tools. Some teachers may try to build a conceptual basis for the algorithms (level 4), but such efforts seem ill-fated if level 2 and 3 understandings are bypassed. Other teachers introduce the standard algorithms at the syntactic level, thereby undermining the development of adequate number concepts and fostering a perspective of mathematics as instrumental understanding. Rather than trying to discourage counting, teachers should help children to become efficient and accurate counters, and help develop level 3 understanding of numeration and computation, i.e. give *much more emphasis* to the first three levels of understanding. Level 3 understanding provides sufficiently powerful computational strategies, so that the introduction of the standard written algorithms may be delayed, if they should be taught at all.

The influence of computing technology necessitates a re-orientation of goals of elementary school mathematics, especially regarding the role of pencil-and-paper computation. There is a call for de-emphasizing standard written algorithms and integrating the calculator into the curriculum as the primary computational tool, accompanied by an increased emphasis on mental methods, estimation, understanding of number and algorithmic thinking as a mathematical process (eg. Olivier, 1988). It seems that the level 1 to 3 understanding of numbers and computational strategies are exactly those that are necessary for developing the skills of mental methods, estimation, and flexible computational pro- res and the understanding of number and numeration. It must be stressed that the istic strategies are not necessarily mental methods, because some children prefer to record at least some portions of their computations. However, the types of strategies formulated by these third graders themselves correspond very closely to the "mental methods" and "street mathematics" described by authors such as Plunkett (1979) and Carraher (1988).

We have outlined a model describing the development of children's understanding of two-digit numeration and computation. Such a model should be complemented by a teaching program to facilitate transition through the different levels of understanding. We are at present implementing an experimental syllabus based on these ideas in eight schools. We shall report the results of the experiment in due course.

## REFERENCES

Barr, D.C. (1978). A Comparison of Three Methods of Introducing Two-Digit Numeration. **Journal for Research in Mathematics Education, 9** (1), 33–43.

Carpenter, T.P. (1980). Heuristic Strategies used to Solve Addition and Subtraction Problems. In R. Karplus (Ed.), **Proceedings of the Fourth International Conference for the Psychology of Mathematics Education** (pp. 317–321). Berkeley, California: University of California.

Carpenter, T.P. & Moser, J.M. (1982). The Development of Addition and Subtraction Problem-Solving Skills. In T.P. Carpenter, J.M. Moser & T.A. Romberg (Eds.), **Addition and Subtraction: A Cognitive Perspective** (pp. 9–24). Hillsdale, NJ: Lawrence Erlbaum Associates.

Carraher, T.N. (1988). Street Mathematics and School Mathematics. In A. Borbás (Ed.), **Proceedings of the Twelfth Annual Conference of the International Group for the Psychology of Mathematics Education** (pp. 1–23). Veszprém, Hungary.

Cooper, R.G. (1984). Early Number Development: Discovering Number Space with Addition and Subtraction. In C. Sophian (Ed.), **Origins of Cognitive Skills** (pp. 157–192). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kamii, C. (1985). **Young Children Reinvent Arithmetic**. New York: Teachers College Press.

Kamii, C. (1986). Place Value: An Explanation of Its Difficulty and Educational Implications for the Primary Grades. **Journal of Research in Childhood Education, 1** (2), 75–85.

Murray, H. (1988). Towards an Understanding of the Two-Digit Numbers: A Theoretical Perspective on Learning Contexts. **South African Journal of Education, 8** (3), 197–202.

Olivier, A.I. (1988). The Future of Pencil-and-Paper Algorithms in the Arithmetic Curriculum. **Pythagoras, 17**, 11–16.

Plunkett, S. (1979). Decomposition and All That Rot. **Mathematics in School, 8** (3), 2–5.

Resnick, L.B. (1983). A Developmental Theory of Number Understanding. In H.P. Ginsburg (Ed.), **The Development of Mathematical Thinking** (pp. 109–151). New York: Academic Press.

Richards, J. & Carter, R. (1982). The Numeration System. In S. Wagner (Ed.), **Proceedings of the Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education** (pp. 57–63). Athens, Georgia: University of Georgia.

Ross, S.H. (1985). **The Development of Children's Place-value Numeration Concepts in Grades Two through Five**. Unpublished doctoral dissertation, University of California, Berkeley.

Steffe, L.P., Von Glasersfeld, E., Richards, J. & Cobb, P. (1983). **Children's Counting Types: Philosophy, Theory and Application**. New York: Praeger Publishers.